Mathematicians and visuo-spatial thinking

Hester Breman (h.breman@maastrichtuniversity.nl)
Faculty of Psychology and Neuroscience
Maastricht University, The Netherlands

Renee Hoekzema (r.s.hoekzema@vu.nl)
Faculty of Science, Vrije Universiteit (VU) Amsterdam
Amsterdam, The Netherlands

Henrik Kragh Sørensen (henrik.kragh@ind.ku.dk)
Department of Science Education
University of Copenhagen, Denmark

Mikkel Willum Johansen (mwj@ind.ku.dk)
Department of Science Education
University of Copenhagen, Denmark

Rainer Goebel (r.goebel@maastrichtuniversity.nl)
Faculty of Psychology and Neuroscience
Maastricht University, The Netherlands

Abstract

Do mathematicians only think in formulas? We surveyed mathematicians' use of mental imagery via the OSIVQ test (n=232) and four open questions (n=222). Combining expert judgement and text mining count of the number of diagrams in 3799 arXiv articles, we investigated whether the amount of visuo-spatial thinking (if any) is related to an article's subdiscipline (as defined by the MSC2020 classification[1]). Finally, we explored the role of visuo-spatial thinking in mathematical research. We conclude that mathematicians are highly visual as measured by their OSIVQ spatial imagery scores. Roughly one third of the mathematicians in our survey used visuo-spatial thinking regularly and one-third frequently. This is not connected to their subdiscipline. Visual representations are important in mathematical practice. As one expert expressed: "images and even movies are continuously running in our minds".

Keywords: mathematical cognition; visual thinking

Introduction

Possibly as a consequence of the 'formalist' attitude advocated by mathematicians like David Hilbert and Bertrand Russell, diagrams almost disappeared from published mathematical papers around the year 1910 (Johansen & Pallavicini, 2022). This formalist attitude is dismissive of the use of visualizations in publications. Over time the attitude changed, and by 2015, 65% of the papers in mathematics contained at least one diagram (Johansen & Pallavicini, 2022). Along similar lines, forty years ago in cognitive psychology "psychologists believed that human reasoning depended on formal rules of inference akin to those of a logical calculus" (Johnson-Laird, 2010). This slowly made place for other views, like connectionism and dynamical systems. Researchers acknowledged that mental simulation seemed to be of importance for reasoning as well (Hegarty, 2004). (Stylianou & Silver, 2004) showed that expert mathematicians used visualization as a tool for exploration before producing a rigorous proof. Building on results by (Stylianou & Silver, 2004) who showed that experts use visual representations more meaningfully than novices, we ask a larger population of mathematicians the use visuo-spatial thinking.

[1] https://zbmath.org/static/msc2020.pdf

Methods

Research questions Our research questions are as follows:

1. Are mathematicians visuo-spatial thinkers?
2. Does a correlation exist between degree of visual thinking and mathematical subdiscipline?
3. What role does visual thinking play in mathematical research?

We operationalised our research questions by inviting 4400 mathematicians to perform the OSIVQ test (Blazhenkova & Kozhevnikov, 2009) and to answer 4 open questions via the Qualtrics platform from Maastricht University. Ethical permission was granted by the Ethical Committee (ERCPN) of the Faculty of Psychology and Neuroscience of Maastricht University under number 245 159 11 2021.

Research question 1

The Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) is a self-report instrument consisting of 45 multiple-choice questions, which assesses individual differences in object imagery, spatial imagery and verbal imagery. Analysis was performed using Python and Matlab scripts.

Research question 2

The visual nature of the participant's subfield was determined by a) text mining of source data of 3799 arXiv papers of September 2020 via the AWS repository using metadata publicly available from Kaggle (Sørensen & Johansen, 2020) b) expert judgment on the visual nature of MSC 2020 disciplines on a scale of non-visual (-1), somewhat visual (0) to visual (1). We performed a Barnard test on 2×2 frequency tables of both ratings and the OSIVQ visuo-spatial scores, binarized into "high" and "low".

Research question 3

Participant's answers to following questions were coded on an ordinal (questions 1-3) or categorical (question 4) scale by 2 experimenters with an inter-rater agreement of minimally 68% using an iterative and partially grounded approach:

1. Mathematician Terence Tao describes in an interview with Heidelberg Laureate Forum how he mentally "moves through a space" to find a solution for a mathematical problem. Do you recognize this, using mental simulation, or imagining the problem visually? Please elaborate.
2. Do you use diagrams, figures or other visual representations in your research articles? Are they distinct from the representations you use for yourself and for communication with your collaborators? Please elaborate.
3. Does visual thinking help you to get insights (A-ha Erlebnis)? If so, do you experience motivation or joy from those visual representations?
4. What motivates you to do (research) mathematics?

Results

Demographics

The invitations resulted in 232 completed OSIVQ questionnaires (31 female, 181 male, 1 non-binary, 19 non-disclosed). 222 participants of this group also answered the four open questions. Many MSC 2020 subdisciplines were represented, for example '35 Partial differential equations', '05 Combinatorics', '11 Number theory', '55 algebraic topology' and '18 Category theory; homological algebra'. The average years of full-time experience in mathematics since start of PhD was 24.15 years (n=220; we discarded responses with more than 80 years experience as being unlikely).

Research question 1: Are mathematicians visuo-spatial thinkers?

The mathematicians' OSIVQ scores on categories 'object', 'spatial' and 'verbal' imagery were 3.03 (SD = 0.61), 3.53 (SD = 0.54) and 3.18 (SD = 0.58), respectively. A one-way ANOVA indicated that the means of the three score groups were significantly different from each other (F=45.31, p=0.0000), whereas t-tests indicated that the spatial scores were significantly higher than the object and verbal scores (spatial $>$ object: T=9.32, p=0.0000; spatial $>$ verbal: T=6.65, p=0.0000).

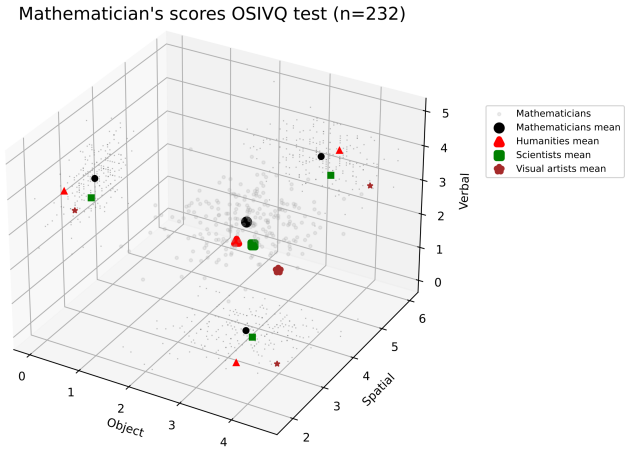

Figure 1: Results of OSIVQ test by mathematicians (n=232) super-imposed on results of (Blazhenkova & Kozhevnikov, 2009). This graph shows that mathematicians (black) on average score higher than researchers in humanities (red), scientists (green) and visual artists (brown) on the spatial axis.

The average score on spatial imagery for mathematicians was M = 3.53 (SD = 0.54); (Blazhenkova & Kozhevnikov, 2009) indicated this was M = 2.63 (SD = 0.66) for researchers in humanities, M = 2.92 (SD = 0.65) for visual artists and M = 3.41 (SD = 0.55) for scientists.

Research question 2: Does an association exist between degree of visual thinking and mathematical subdiscipline?

The frequency tables of low and high visuo-spatial OSIVQ scores vs. low and high visual degree of mathematical subdisciplines, measure by number of diagrams and expert opinion, are shown Table 1. The result of the Barnard test for independence of Tables 1(A) (diagram-based) and (B) (expert-based) was not significant (test statistic=-0.28, p-value=1.0 for (A), test statistic=0.32, p=0.73 for (B)). Similarly we found that OSIVQ object and verbal scores do not correlate with either measure of visual degree of the disciplines.

Table 1: Frequency table of participant's visuo-spatial OSIVQ scores (columns) vs visual nature of discipline

	Visual degree of MSC 2020 discipline			
	(A) Diagram-based		**(B) Expert-based**	
Visuo-spatial OSIVQ score	Low	High	Low	High
Low	60	47	65	42
High	62	45	62	45

Research question 3: What role does visual thinking play in mathematical research?

Results on question 1 were that one group indicated not to use visual mental imagery (n=58), one group sometimes uses

mental imagery (n=55) and one group of respondents is using mental imagery very frequently (n=79), with inter-rater agreement of 68%.

Regarding question 2, 68% of respondents state that they use visualization in papers with an inter-rater agreement of 75%.

For 68% of participants visual thinking helps to get insights with 91% inter-rater agreement.

The most important motivation (56%) for the participants to do research mathematics as answered to question 4 was "Direct emotional satisfaction connected to the activity (e.g. pleasure, challenge, competition, interesting, curiosity, fun, passion, puzzle solving)", followed by the "special nature of mathematics" (15%). The raters agreed for 70%.

Conclusion and discussion

Even though a large portion of the articles (1846 of 3799) sampled from arXiv have no figure environments, visuo-spatial thinking turns out to be important to mathematicians, with roughly one third using visuo-spatial thinking regularly and one-third frequently. Further research would be desirable to explore the role of visualization in mathematical thinking, for the benefit of the mathematical cognition research agenda (Alcock et al., 2016) and education (Gilligan-Lee, Hawes, & Mix, 2022).

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References

Alcock, L., Ansari, D., Batchelor, S., Bisson, M., Smedt, B. D., Gilmore, C., ... Weber, K. (2016). Challenges in mathematical cognition a collaboratively-derived research agenda. *Journal of Numerical Cognition*.

Blazhenkova, O., & Kozhevnikov, M. (2009). The new object-spatial-verbal cognitive style model: Theory and measurement. *Applied Cognitive Psychology*, *23*, 638–663.

Gilligan-Lee, K., Hawes, Z., & Mix, K. (2022). Spatial thinking as the missing piece in mathematics curricula. *npj Science of Learning*.

Hegarty, M. (2004). Mechanical reasoning by mental simulation. *Trends in Cognitive Sciences*, *8*, 280–285.

Johansen, M., & Pallavicini, J. (2022). Entering the valley of formalism: trends and changes in mathematicians' publication practice—1885 to 2015. *Synthese*.

Johnson-Laird, P. (2010). Mental models and human reasoning. *Proc Natl Acad Sci U.S.A.*, *107*(43), 18243–18250.

Sørensen, H., & Johansen, M. (2020). Counting mathematical diagrams with machine learning. In *Diagrams 2020, lnai 12169.*

Stylianou, D., & Silver, E. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, *6*(4), 353–387.